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BARRIER OPTIMIZATION ALGORITHM APPLIED TO CALCULATION OF OPTIMAL LOADING OF DISSIMILAR CABLES IN ONE TRENCH

Wael Moutassam, University of Toronto, (Canada), Email: wael.moutassam@utoronto.ca
George J. Anders, Kinectrics Inc., (Canada), Email: george.anders@attglobal.net

ABSTRACT

This paper presents an optimization algorithm applied to the rating calculation of unequally loaded electric power cables. Whereas standards spell out the principles of rating calculations for single and identical equally loaded cables, the common situation where the cables in a trench are of different construction is given only a scant treatment. The paper does not introduce any new calculation method but addresses an issue of what is the best method of loading groups of cables in a common trench. To answer this question, an optimization algorithm utilizing barrier method is introduced and its application illustrated for a complex practical cable arrangement.

KEYWORDS

power cables, rating calculations, unequally loaded cables, barrier algorithm.

Author Name & Affiliation

Wael MOUTASSAM, University of Toronto, (Canada), wael.moutassam@utoronto.ca
George ANDERS, Kinectrics Inc., (Canada), george.anders@attglobal.net

INTRODUCTION

Information about the maximum current-carrying capacity which a cable can tolerate throughout its life without risking deterioration or damage is extremely important in power cable engineering and operation. Ampacity values are required for every new cable installation, as well as for cable systems in operation. With some underground transmission cable circuits approaching the end of their design life, the development of a systematic method for determining the feasibility of extending cable life and/or increasing current ratings is of paramount importance.

Current rating techniques of electric power cables have as long a history as the cable itself. Methods presented by Kennelly in 1893, [1] and Mie in 1905, [2], are still used in today's standards. Over the last hundred years many researchers and engineers have worked on various aspects of cable ratings and several international standards are in use today based on these works [3-6]. It would seam that not much new can be said about rating calculation methods. However, even today the work on refining cable ampacity computations is being continued. It proceeds in several directions: 1) experimental studies are being performed to fine-tune some of the computational formulas and adjust the value of constants, 2) numerical methods are being applied to overcome limitations inherent in the analytical approach, 3) computational methods are being developed for rating calculations for cables laid in non standard conditions (e.g., deep tunnels, ventilated troughs or ducts filled with water) and 4) real time rating algorithms are being developed. The developments presented in this paper fall into the first category above. They constitute an incremental improvement in the power cable ampacity calculation methods addressing an issue of loading of different cables types laid in the same trench.

Analysis of unequally loaded/dissimilar cables is given only a scant attention in the published literature. The method proposed in the IEC standard [3], whose mathematical basis is discussed in [7], outlines a procedure for calculation of the influence of the neighboring cables on the rating of the cable of interest. This procedure is a starting point for the developments presented in this paper and is summarized in Chapter II. The procedure is iterative in nature considering one cable at the time and adjusting its rating based on the loading of the remaining cables in the group. This way, a solution to the ampacity problem can be obtained, which although satisfactory, may not lead to the optimal cable utilization. In order to find an optimal solution for the problem of loading of dissimilar cables an optimization problem is formulated in Chapter III. Application of the algorithm is demonstrated in Chapter IV which contains also comparative studies of the proposed solution with that obtained using the method from the standards. A summary section closes the presentation.

RATING OF UNEQUALLY LOADED/DISSIMILAR CABLES

This section presents an overview and the method of rating calculations described in the IEC Standard 60287. It will be a starting point to the developments presented in the following chapters.

Cable rating equations

Steady-state rating computations involve solving the equation for the ladder network with the thermal capacitances neglected [7]. The unknown quantity is either the conductor current *I* (A) or its operating temperature θ_c (°C). In the first case, the maximum operating conductor temperature is given, and in the second case, the conductor current is specified. In this paper, our interest is in finding conductor current, hence the following rating equation will be used for the cable with *n* conductors [3].

$$I = \left[\frac{\Delta\theta - W_d\left[0.5T_1 + n(T_2 + T_3 + T_4)\right]}{RT_1 + nR(1+\lambda_1)T_2 + nR(1+\lambda_1+\lambda_2)(T_3+T_4)} \right]^{0.5} \quad (1)$$

where

$\Delta\theta = \theta_c - \theta_{amb}$, the latter representing ambient temperature.

W_d (W/m) represents dielectric losses per unit length.

λ_1 , λ_2 are the sheath and armor loss factors.

R is the ac resistance per unit length of the conductor at maximum operating temperature (ohm/m).

T_1, T_2, T_3 and T_4 are the thermal resistances where T_1 is the thermal resistance per unit length between one conductor and the sheath, T_2 is the thermal resistance per unit length of the bedding between sheath and armor, T_3 is the thermal resistance per unit length of the external serving of the cable, and T_4 is the thermal resistance per unit length between the cable surface and the surrounding medium, (Km/W).

The expression for the external thermal resistance of an isolated cable is given by

$$T_4 = \frac{\rho_s}{2\pi}\ln\left(u + \sqrt{u^2 - 1}\right) \quad (2)$$

where ρ_s (Km/W) is the thermal resistivity of the soil and $u = 2L/D_e$ with D_e = external diameter of the cable, (mm) and L = depth of burial of the centre of the cable, (mm).

Standard treatment of unequally loaded cables

The method suggested for the calculation for ratings of a group of cables set apart is to calculate the temperature rise at the surface of the cable under consideration caused by the other cables of the group, and to subtract this rise from the value of $\Delta\theta$ used in the equation (1) for the rated current. An estimate of the power dissipated per unit length of each cable must be made beforehand, and this can be subsequently amended as a result of the calculation where it becomes necessary.

The temperature rise $\Delta\theta_{kp}$ at the surface of the cable *p* produced by the power W_k watt per unit length dissipated in cable *k* is equal to

$$\Delta\theta_{kp} = \frac{\rho_s}{2\pi} W_k \ln\frac{d'_{pk}}{d_{pk}} \quad (3)$$

The distances of d_{pk} and d'_{pk} are measured from the centre of the p^{th} cable to the centre of cable *k*, and the centre of the reflection of cable *k* in the ground-air surface, respectively (see Fig. 1). Thus, the temperature rise $\Delta\theta_p$ above ambient at the surface of the p^{th} cable, whose rating is being determined, caused by the power dissipated by the other (*q*-1) cables in the group, is given by

$$\Delta\theta_p = \Delta\theta_{1p} + \Delta\theta_{2p} + \ldots + \Delta\theta_{kp} + \ldots + \Delta\theta_{qp} \quad (4)$$

with the term $\Delta\theta_{pp}$ excluded from the summation.

The value of $\Delta\theta$ in (1) for the rated current is then reduced by the amount of $\Delta\theta_p$ and the rating of the p^{th} cable is determined using a value of T_4 corresponding to an isolated cable at position *p*. That is,

$$I_p = \left[\frac{\Delta\theta - W_d\left[0.5T_1 + n(T_2 + T_3 + T_4)\right] - \Delta\theta_p}{RT_1 + nR(1+\lambda_1)T_2 + nR(1+\lambda_1+\lambda_2)(T_3+T_4)} \right]^{0.5} \quad (5)$$

with all parameters in this equation pertaining to cable *p*.

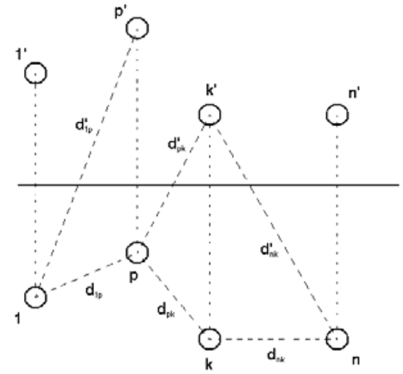


Fig. 1 Cables buried underground and their images with respect to the earth's surface

This calculation is performed for all cables in the group, and is repeated where necessary to avoid the possibility of overheating any of the cables.

Let θ_{ep} denote the external temperature of cable *p* in isolation. Substituting (3) into the right-hand side of (4) and applying equation (2), the following general expression for the external thermal resistance of cable *p* is obtained:

$$T_4^p = \frac{\theta_{ep} + \Delta\theta_p - \theta_{amb}}{W_p} = \frac{\rho_s}{2\pi}\left(\ln\left(u + \sqrt{u^2 - 1}\right) + \frac{1}{W_p}\sum_{\substack{k=1 \\ k \neq p}}^{q} W_k \ln\frac{d'_{pk}}{d_{pk}} \right)$$

(6)

When a group of identical, equally loaded cables is considered, the computations can be much simplified. In this type of grouping, the rating of the group is determined by the ampacity of the hottest cable. It is usually possible to decide from the configuration of the installation which cable will be the hottest, and to calculate the rating for this one. In cases of difficulty, a further calculation for another cable may be necessary. The method is to calculate a modified value of T_4 which takes into account the mutual heating of the group and to leave unaltered the value of $\Delta\theta$ used in the rating equation (1).

When the losses in the group of cables are equal, equation (6) simplifies to

$$T_4 = \frac{\rho_s}{2\pi}\ln\left\{\left(u + \sqrt{u^2 - 1}\right)\cdot\left[\left(\frac{d'_{p1}}{d_{p1}}\right)\left(\frac{d'_{p2}}{d_{p2}}\right)\cdots\left(\frac{d'_{pk}}{d_{pk}}\right)\cdots\left(\frac{d'_{pq}}{d_{pq}}\right)\right]\right\} \quad (7)$$

There are (q-1) factors in square brackets, with the term $\frac{d'_{pp}}{d_{pp}}$ excluded.

FORMULATION OF THE OPTIMIZATION PROBLEM

This section describes the proposed solutions for optimal loading of dissimilar cables. Let us consider the system of 'q' cables. Example of such system is shown in Fig. 2:

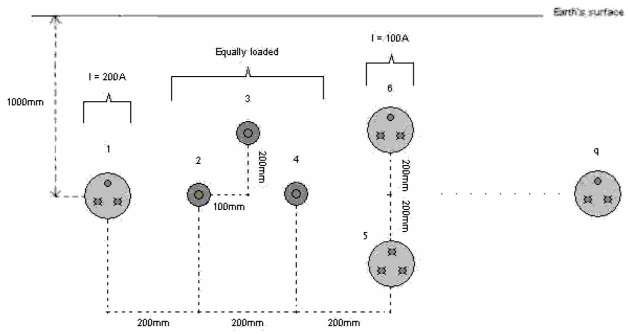


Fig. 2: General system including 'q' cables. Some cables are equally loaded, some are unequally loaded and some have preset fixed currents.

Let's assume that the operating temperatures of the cables are $\theta_1, \theta_2, ..., \theta_q$ and the maximum allowed operating temperatures are $\theta_{1\max}, \theta_{2\max}, ..., \theta_{q\max}$. The conductors' temperature rises above ambient temperature are $\Delta\theta_1, \Delta\theta_2, ..., \Delta\theta_q$, where $\Delta\theta_i = \theta_i - \theta_{amb}$. Also, let's assume that the cables are laid in a uniform medium; system of cables laid in a non uniform medium can be treated by modifying the value of T_4 [7]. By calculating the ampacities of a group of cables we are trying to find the maximum currents that can be carried by all cables without any of the cable temperatures exceeding its maximum allowed value. This may be expressed mathematically as an optimization problem as shown below.

$$\begin{aligned} &\textit{Maximize} && I_1 + I_2 + \cdots + I_n \\ &\textit{subject to} && \theta_1 \le \theta_{1\max},\ \theta_2 \le \theta_{2\max}, ..., \theta_q \le \theta_{q\max} \\ &&& I_{ia} = I_{ib} = I_{ic};\ i = 1, ..., k \end{aligned} \tag{8}$$

where k is the number of circuits composed of single core cables per phase. The last constraint states that the system is balanced.

All constraints in (8) will now be modified to be expressed as functions of the currents. For the cable *p*, we have $\theta_p \le \theta_{p\max}$ and therefore, $\Delta\theta_p \le \Delta\theta_{p\max}$. Thus, using (5), we can write

$$I_p^2 \le \frac{\Delta\theta_{p\max} - W_{dp}\left[0.5T_{1p} + n_p(T_{2p} + T_{3p} + T_{4p})\right] - \Delta\theta_p}{R_pT_{1p} + n_pR_p(1+\lambda_{1p})T_{2p} + n_pR_p(1+\lambda_{1p}+\lambda_{2p})(T_{3p}+T_{4p})} \tag{9}$$

Expressing W_k in (3) in terms of the current I_k and then substituting (3) and (4) into (9), we obtain

$$\frac{c_{p1}}{d_p}I_1^2 + \frac{c_{p2}}{d_p}I_2^2 + \ldots + \frac{1}{d_p}I_p^2 + \ldots + \frac{c_{pq}}{d_p}I_q^2 \le 1 \tag{10}$$

where

$$c_{pj} = \frac{n_jR_j(1+\lambda_{1j}+\lambda_{2j})\mu_j \frac{\rho_s}{2\pi}\ln\frac{d'_{pj}}{d_{pj}}}{R_pT_{1p} + n_pR_p(1+\lambda_{1p})T_{2p} + n_pR_p(1+\lambda_{1p}+\lambda_{2p})(T_{3p}+T_{4p})} \tag{11}$$

$$d_p = \frac{\Delta\theta_{p\max} - W_{dp}\left[0.5T_{1p} + n_p(T_{2p}+T_{3p}+T_{4p})\right]}{R_pT_{1p} + n_pR_p(1+\lambda_{1p})T_{2p} + n_pR_p(1+\lambda_{1p}+\lambda_{2p})(T_{3p}+T_{4p})}$$

$$- \frac{\frac{\rho_s}{2\pi}\left[\sum_{\substack{j=1 \\ j\ne p}}^{q}\left(n_jW_{dj}\ln\frac{d'_{pj}}{d_{pj}}\right)\right]}{R_pT_{1p} + n_pR_p(1+\lambda_{1p})T_{2p} + n_pR_p(1+\lambda_{1p}+\lambda_{2p})(T_{3p}+T_{4p})} \tag{12}$$

In what follows we will rewrite the optimization problem (8) without the last equality constraint, which is understood to be enforced. The optimization problem (8) can now be rewritten as

$$\begin{aligned} &\textit{Minimize} && -I_1 - I_2 - \cdots - I_q \\ &\textit{subject to} && \frac{1}{d_1}I_1^2 + \frac{c_{12}}{d_1}I_2^2 + \ldots + \frac{c_{1q}}{d_1}I_q^2 \le 1 \\ &&& \frac{c_{21}}{d_2}I_1^2 + \frac{1}{d_2}I_2^2 + \ldots + \frac{c_{2q}}{d_2}I_q^2 \le 1 \\ &&& \vdots \\ &&& \frac{c_{p1}}{d_p}I_1^2 + \frac{c_{p2}}{d_p}I_2^2 + \ldots + \frac{1}{d_p}I_p^2 + \ldots + \frac{c_{pq}}{d_p}I_q^2 \le 1 \\ &&& \vdots \\ &&& \frac{c_{q1}}{d_q}I_1^2 + \frac{c_{q2}}{d_q}I_2^2 + \ldots + \frac{1}{d_q}I_q^2 \le 1 \end{aligned} \tag{13}$$

In a matrix form this can be expressed as shown in (14). Solution to this optimization problem is discussed in the Appendix.

$$\begin{aligned} &\textit{Minimize} && -I_1 - I_2 - \cdots - I_q \\ &\textit{subject to} && \begin{bmatrix} \frac{1}{d_1} & \frac{c_{12}}{d_1} & \cdots & \cdots & \frac{c_{1q}}{d_1} \\ \frac{c_{21}}{d_2} & \frac{1}{d_2} & \frac{c_{23}}{d_2} & \cdots & \frac{c_{2q}}{d_2} \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \frac{c_{q-1,q}}{d_{q-1}} \\ \frac{c_{q1}}{d_q} & \cdots & \cdots & \frac{c_{q,q-1}}{d_q} & \frac{1}{d_q} \end{bmatrix} \begin{bmatrix} I_1^2 \\ \vdots \\ \vdots \\ \vdots \\ I_q^2 \end{bmatrix} \le \begin{bmatrix} 1 \\ \vdots \\ \vdots \\ \vdots \\ 1 \end{bmatrix} \end{aligned} \tag{14}$$

APPLICATION OF THE BARRIER METHOD

The optimization problem (14) can be written in the following form:

$$\begin{aligned} &\textit{Minimize} && f_0(x) \\ &\textit{subject to} && f_i(x) \le 0 \quad i = 1, \ldots, n \end{aligned} \tag{15}$$

where $x = [I_1, ..., I_q]^T$

First, we will need to check whether the problem is convex. The Hessian of the objective function is equal to:

$$\nabla^2 f_0(x) = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \tag{16}$$

and is non-negative. Therefore, the objective function is convex. The Hessian of the i^{th} inequality constraint is given by

$$\nabla^2 f_i(x) = \begin{bmatrix} \frac{2c_{i1}}{d_i} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \vdots & 0 & \frac{2c_{in}}{d_i} \end{bmatrix} \qquad (17)$$

First we observe that by definition c_{ij} is always positive. On the other hand, d_i is also positive in spite of the subtraction in the numerator. The reason for this is that the terms that are being subtracted represent the temperature increase in the conductor due to dielectric losses of the ith cable and the remaining cables in the same trench, which are in the order of a few degrees, usually very small values. Therefore, the Hessian above is non-negative. Thus, all inequality constraint functions are convex and the optimization problem (14) is convex.

The optimization problem (14) is equivalent to the following one [10]

$$Minimize \quad f_0(x) + \sum_{i=1}^{n} I\_(f_i(x)) \qquad (18)$$

The indicator function *I_* is defined by

$$\hat{I}\_(u) = -\frac{1}{t}\log(-u) \qquad (19)$$

where t > 0 is a parameter that sets the accuracy of the approximation.

$\hat{I}\_$ is convex and non-decreasing and takes on the value ∞ for u > 0. $\hat{I}\_$ is also differentiable and closed. As t increases the approximation becomes more accurate, as shown in the graph below [10].

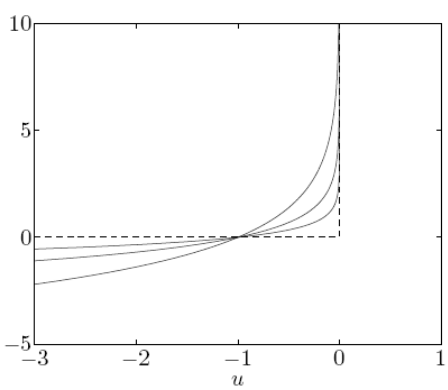


Fig. 6: Dashed line shows $I\_$ whereas the solid lines show $\hat{I}\_(u)$ for t = 0.5, 1, 2. A larger value of t provides a better approximation.

The function,

$$\phi = -\sum_{i=1}^{n} \log(-f_i(x)) \qquad (20)$$

is called the *logarithmic barrier* for problem (18). Its domain contains the set of values, x that satisfy the inequality constraints, $f_i(x) \le 0$, strictly, i.e. $f_i(x) < 0$.

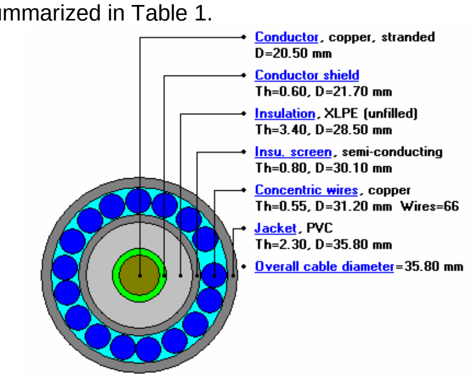
Therefore (18) becomes,

$$Minimize \quad f_0(x) + \left(\frac{1}{t}\right)\phi \qquad (21)$$

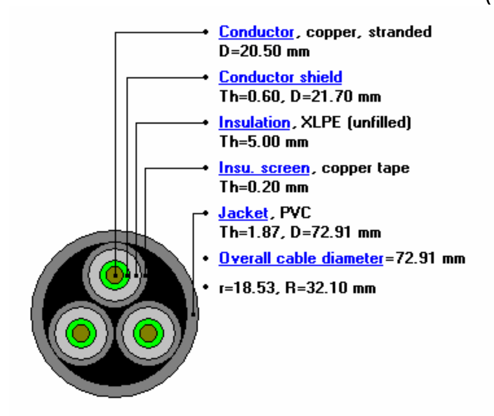
The algorithm used to solve this problem is described in the Appendix.

COMPARATIVE STUDIES

The numerical studies presented in this section will illustrate the efficacy of the proposed algorithm. Two computational procedures will be used. One, based on the IEC 60287 iterative procedure method outlined in Section II and the other using the optimization algorithm presented in Section III. Two different cable types will be used in this analysis as shown in Figure 3 and the computed parameters summarized in Table 1.



(a)



(b)

Fig 3. Cables considered in the numerical example. (a) 10 kV-300 mm² single core, single point bonded circuit (b) 10 kV-300 mm² three-core. The graphs obtained with a CYMCAP [8] program.

Table 1 Computed parameters in the numerical example (non zero values shown)

Computed parameter	Single core cable	Three core cable
T_1 $(K.m/W)$	0.214	0.325
T_2 $(K.m/W)$	0.109	0.042
T_4 $(K.m/W)$ *	0.751	0.637
R (Ω/km) at $90^{\circ}C$	0.0763	0.0790

*The external thermal resistance is for a single circuit in isolation with the cable 1m underground, soil thermal resistivity of 1 Km/W and soil ambient temperature of 15°C.

We will consider a system composed of 3 circuits with 6 cables as shown in Fig.4. Cables 1 and 6 carry fixed currents of 200 A and 100 A, respectively.

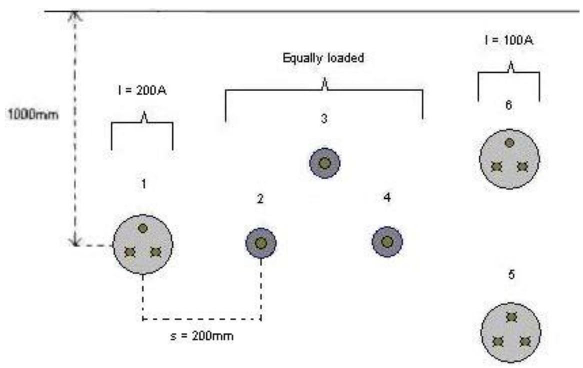


Fig. 5 Six cables with two circuits with fixed current.

The results are summarized in Table 3.

Table 3 Comparative results for dissimilar cables

IEC Iterative approach		**Optimization**	
Ampacity (A)	Conductor Temperature (^oC)	Ampacity (A)	Conductor Temperature (^oC)
$I_1 = 200$	$\theta_1 = 56.3$	$I_1 = 200$	$\theta_1 = 59.6$
$I_2 = 564$	$\theta_2 = 77.0$	$I_2 = 667$	$\theta_2 = 88.0$
$I_3 = 564$	$\theta_3 = 72.0$	$I_3 = 667$	$\theta_3 = 83.2$
$I_4 = 564$	$\theta_4 = 81.1$	$I_4 = 667$	$\theta_4 = 90.0$
$I_5 = 527$	$\theta_5 = 90.0$	$I_5 = 405$	$\theta_5 = 74.7$
$I_6 = 100$	$\theta_6 = 53.4$	$I_6 = 100$	$\theta_6 = 54.3$
Total 2519 A		**Total 2705 A**	

We can observe that, in this case, the optimization algorithm gives a total ampacity which is *186A* (7.4%) greater than the total ampacity obtained with the IEC method. Another interesting observation is that the ampacities of cables 2 to 5 (cables with non fixed currents) are much closer to each other for the IEC case than for the optimization case.

To test the accuracy of the algorithm presented here, a commercial optimization problem solver called KNITRO [9] was used to find the solution to the last optimization problem. The same results were obtained with both algorithms.

CONCLUSIONS

This paper presented an optimization algorithm for the analysis of loading of dissimilar cables. The proposed approach applies a Barrier Optimization method to solve an optimization problem with linear objective function and quadratic constraints. Numerical results were obtained for two cases, one with identical, but unequally loaded cables, and the other with dissimilar cables. In each case, the ampacity computed with the new algorithm was higher than the results obtained using an iterative approach.

There is one additional advantage of the proposed method in comparison with the one recommended by the standards. In the standard method the result depend on the selection of the starting cable or the reference cable and is fairly difficult to implement whereas the optimization formulation is clear and easy to program.

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APPENDIX

Algorithms for the Barrier Optimization Method

Introduction

We recall that the optimization problem we are solving is given by (21). Since (21) is now obtained through approximating the indicator function $I_{\_}$ by $\hat{I}_{\_}$ then its solution will be an approximation to the solution of the original problem (15). However, as *t* increases this approximate solution improves. Newton’s method [10] will be used to solve (21). Unfortunately for very large *t*, the objective function in (21) is difficult to minimize using the Newton’s method since its Hessian varies rapidly near the boundary of the feasible set. This problem can be mitigated by solving a sequence of problems, each having the form of (21), starting with small *t* and increasing it at each step. The solution at each step is used as the initial condition for the next step. As *t* grows very large the approximate solution becomes very accurate [10].

This method of approximating the indicator function by a logarithmic barrier and solving the resulting optimization problem through a sequence of steps is called the *barrier method* [10].

Algorithms

The algorithm for the barrier method is given below [10].

Given: strictly feasible x, $t = t^{(0)}, \mu > 1$, tolerance $\varepsilon_{out} > 0$

Repeat: 1. Compute the solution of $Minimize\ f_0(x) + \left(\frac{1}{t}\right)\phi$, $x^*(t)$, starting at x, using the Newton's method.

2. Update $x = x^*(t)$
3. Stopping criterion: stop if $n/t < \varepsilon_{out}$
4. Increase $t = \mu.t$

Each iteration involving increasing t and resolving the problem is referred to as an outer iteration. Each iteration within the Newton's method is referred to as an inner iteration. Since μ is the factor by which t is increased, the choice of μ will determine how many outer iterations there will be. Based on experience with the algorithm a choice of this parameter that results in good convergence speed is $\mu = 50$, [11]. More details are given in the following sections.

Barrier Method (Including Newton's Method) Algorithm

Newton's method is used for finding the solution of (21) at each step (or outer iteration) of the barrier method algorithm. Including the Newton's algorithm in the barrier method gives the following algorithm.

Given: strictly feasible x, $t = t^{(0)}, \mu > 1$, tolerance $\varepsilon_{out} > 0$

Repeat: 1. Compute the solution $x^*(t)$ of (21) starting at x, using Newton's method :

Given: starting feasible point x^*,($x^* = x$), tolerance $\varepsilon_{in} > 0$, $\alpha \in \left(0, \frac{1}{2}\right), \beta \in (0,1)$.

Repeat: a. Compute Δx^* by solving the matrix equation:

$$\left[\nabla^2 f_0(x^*) + \frac{1}{t}\nabla^2\phi(x^*)\right]\left[\Delta x^*\right] = \underbrace{\left[-\nabla f_0(x^*) - \frac{1}{t}\nabla\phi(x^*)\right]}_{r(x^*)}$$

b. Compute damping factor t_{in} for updating x^*:

$t_{in} = 1$

$while\ \ norm\left(r\left(x^* + t_{in}\Delta x^*\right)\right) \geq (1 - \alpha \cdot t_{in}) \cdot norm\left(r\left(x^*\right)\right)$

$t_{in} = \beta \cdot t_{in}$

c. Update $x^* = x^* + t_{in}\Delta x^*$.

d. Stopping criterion: stop if $norm\left(r\left(x^*\right)\right) \leq \varepsilon_{in}$.

2. Update $x = x^*(t)$
3. Stopping criterion: stop if $n/t < \varepsilon_{out}$
4. Increase $t = \mu \cdot t$

In the above algorithm, the *norm* function is the Euclidean norm. ∇ is the gradient operator and ∇^2 is the Hessian operator. Δx is used in updating x. The variation in x due to adding Δx to it is dampened by the damping factor, t_{in}. α is used in the condition statement for finding a suitable t_{in}. β is the factor used in decreasing the value of t_{in} to a suitable value. Based on experience with the algorithm, good choices for α and β that result in suitable values for t_{in} are 0.3 and 0.7, respectively.

The resulting solution x from the *barrier method* algorithm will be a very good approximate solution to the original optimization problem.

Calculating Gradients and Hessians

In step 1.a. in the above algorithm the vectors $\nabla f_0(x^*)$ and $\nabla\phi(x^*)$ and matrices $\nabla^2 f_0(x^*)$ and $\nabla^2\phi(x^*)$ are needed in order to compute Δx^*. The following shows how these vectors and matrices may be calculated for the optimization problem (21).

The gradient of $f_0(x)$ evaluated at x^* is $\nabla^2 f_0(x^*)$ and is given by:

$$\nabla f_0(x^*) = [-1, -1, ..., -1]^t a \tag{22}$$

where $f_0(x) = -I_1 - I_2 - \ldots - I_q$, and $x = \left[I_1, I_2, ..., I_q\right]^t$.\`

The gradient of $\phi(x)$ evaluated at x^* is $\nabla\phi(x^*)$ and is given by:

$$\nabla\phi(x^*) = \left.\begin{bmatrix} \sum_{i=1}^{q} \frac{1}{f_i(x)} \cdot \frac{\partial f_i}{\partial x_1} \\ \sum_{i=1}^{q} \frac{1}{f_i(x)} \cdot \frac{\partial f_i}{\partial x_2} \\ \vdots \\ \sum_{i=1}^{q} \frac{1}{f_i(x)} \cdot \frac{\partial f_i}{\partial x_q} \end{bmatrix}\right|_{x=x^*}$$

$$= \left.\begin{bmatrix} -2x_1 \cdot \left[\frac{1}{f_1(x)} \cdot \frac{1}{d_1} + \frac{1}{f_2(x)} \cdot \frac{c_{21}}{d_2} + \ldots + \frac{1}{f_q(x)} \cdot \frac{c_{q1}}{d_q}\right] \\ -2x_2 \cdot \left[\frac{1}{f_1(x)} \cdot \frac{c_{12}}{d_1} + \frac{1}{f_2(x)} \cdot \frac{1}{d_2} + \ldots + \frac{1}{f_q(x)} \cdot \frac{c_{q2}}{d_q}\right] \\ \vdots \\ -2x_q \cdot \left[\frac{1}{f_1(x)} \cdot \frac{c_{1n}}{d_1} + \frac{1}{f_2(x)} \cdot \frac{c_{2n}}{d_2} + \ldots + \frac{1}{f_q(x)} \cdot \frac{1}{d_q}\right] \end{bmatrix}\right|_{x=x^*} \tag{23}$$

The Hessian of $f_0(x)$ evaluated at x^* is $\nabla^2 f_0(x^*) = \mathbf{0}$. The Hessian of $\phi(x)$ evaluated at x^* is $\nabla^2\phi(x^*)$ and is given by:

$$\nabla^2\phi(x^*) = \left.\begin{bmatrix} \frac{\partial}{\partial x_1}\left(\sum_{i=1}^{q} \frac{1}{f_i(x)} \cdot \frac{\partial f_i}{\partial x_1}\right) & \cdots & \frac{\partial}{\partial x_q}\left(\sum_{i=1}^{q} \frac{1}{f_i(x)} \cdot \frac{\partial f_i}{\partial x_1}\right) \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_1}\left(\sum_{i=1}^{q} \frac{1}{f_i(x)} \cdot \frac{\partial f_i}{\partial x_q}\right) & \cdots & \frac{\partial}{\partial x_q}\left(\sum_{i=1}^{q} \frac{1}{f_i(x)} \cdot \frac{\partial f_i}{\partial x_q}\right) \end{bmatrix}\right|_{x=x^*}$$

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